# DASP3rd Chapter 4 - Exercises

## 1 Oversampling

1. How do we define the power spectral density (PSD) $S_{XX}(e^{j\Omega})$ of a signal $x(n)$?

   Solution:

   - Measure of how the power in a sequence $x(n)$ is distributed over frequency
   - Defined as Fourier Transform of the auto-correlation sequence $r_{XX}(n)$ of the discrete-time sequence given by

   $$S_{XX}\left(e^{j\Omega}\right) = \sum_{n=-\infty}^{\infty} r_{XX}(n) \cdot e^{-j\Omega n}.$$

2. What is the relationship between signal power $\sigma_X^2$ (variance) and power spectral density $S_{XX}(e^{j\Omega})$?

   Solution:

   - The relationship is given by PARSEVAL THEOREM
     (actually this is a special case of the WIENER-KHINTCHINE THEOREM for $n = 0$)

   $$\sigma_X^2 = \sum_{n=-\infty}^{\infty} (x(n))^2 = r_{XX}(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} S_{XX}\left(e^{j\Omega}\right) d\Omega$$

   ⇒ Energy of a signal in time domain = energy in frequency domain!

3. Why do we need to oversample a time-domain signal?

   Solution:

   - To avoid aliasing in the frequency domain. As an example some non-linear operations generate harmonics that go above the Nyquist frequency. Oversampling can avoid that problem by increasing temporarily the sampling frequency.

4. Explain why an oversampled pulse-code modulation (PCM) AD converter has lower quantization noise power in the baseband than a Nyquist-rate sampled PCM AD converter?

   Solution:

   - PSD of quatization error

   $$S_{EE}(f) = \frac{Q^2}{12 f_S} \qquad \rightarrow \text{see Book, Eq. (4.1)}$$

   - For Nyquist sampling $f_S = 2f_B$ the noise power in audio band leads to

   $$N_B^2 = \sigma_E^2 = 2 \cdot \int_0^{f_B} S_{EE}(f) df = 2 \cdot \frac{Q^2}{12 \cdot 2 f_B} \cdot [f_B - 0] \qquad \rightarrow \sigma_E^2 = \frac{Q^2}{12}.$$

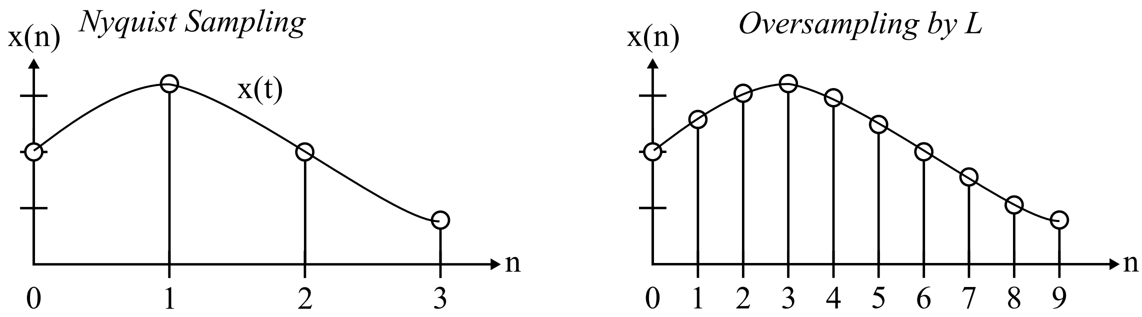
- For oversampling by factor $L \rightarrow f_S = L \cdot 2f_B$:

$$N_B^2 = \sigma_E^2 = 2 \cdot \frac{Q^2}{12 \cdot L \cdot 2f_B} \cdot [f_B - 0] \qquad \rightarrow \sigma_E^2 = \frac{Q^2}{12L}$$

  $\rightarrow$ Noise power in the audio band is $L$-times lower!

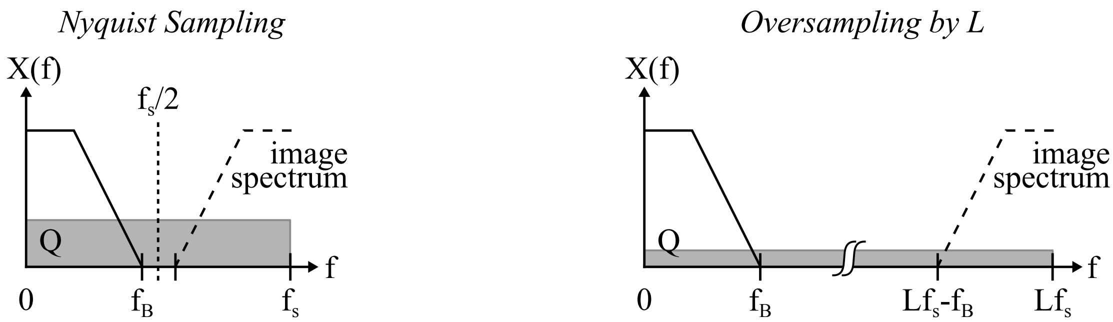
5. How do we perform oversampling by a factor of $L$ in the time domain?

   Solution: $L = 3$




6. Explain the frequency domain interpretation of the oversampling operation.

   Solution:




7. What is the passband and stopband frequency of the analog anti-aliasing filter?

   Solution:

   - Humans are able to perceive frequencies <u>up to 20 kHz</u>
     * $f_B = 20$ kHz, $\rightarrow f_{pass} = f_B$ (passband)
     * $f_{stop} = f_S - \frac{f_S}{2} = \frac{f_S}{2}$ (stopband)
     * $f_{trans} = f_{stop} - f_{pass} = \frac{f_S}{2} - f_B = 2.05\,\text{kHz}$ for $f_S = 44.1\,\text{kHz}$ (transition band)
     * To avoid aliasing, an analog filter with high slope is required (narrow transition band)!
   - Using oversampling the analog filter can have lower order!
     * $f_B = 20$ kHz, $\rightarrow f_{pass} = f_B$ (passband, remains the same)
     * $f_{stop} = L \cdot f_S - \frac{f_S}{2}$ (stopband)
     * $f_{trans} = f_{stop} - f_{pass} = L \cdot f_S - \frac{f_S}{2} - f_B$ (large transition band)

8. What is the passband and stopband frequency of the digital anti-aliasing filter before downsampling?

   Solution:

   - Passband remains $f_{pass} = f_B$
   - Stopband changes $f_{stop} = \frac{f_S}{2}$

9. How is the downsampling operation performed (time domain and frequency domain explanation)?

   Solution:

   - Downsampling by factor $L$:
     * Time domain: take every $L^{th}$ sample
     * Frequency domain: image spectrum is moved from $[L{\cdot}f_S - \frac{f_S}{2}, L{\cdot}f_S]$ to $[f_S - \frac{f_S}{2}, f_S]$, which is explained by Eq. (3.17) in Chapter 3.

# 2 Delta-sigma Conversion

1. Why can we apply noise shaping in an oversampled AD converter?

   Solution:

   - The quantization noise in an oversampled AD converter is spread from 0 up to $Lf_S$. This oversampling allows the noise shaping applied to the quantizer. Thus the quantization noise in the audio band is suppressed and shifted to the frequency region above 20 kHz.

2. Show how the delta-sigma converter (DSC) has a lower quantization noise in the audio band than an oversampled PCM AD converter?

   Solution:

   - Let us consider the variance for the input and the quantization error after oversampling to be defined as $\sigma_X^2 = \left(\frac{x_{max}}{P_F}\right)^2$, and $\sigma_E^2 = \frac{Q^2}{12L}$.
   One can see from the formula that the PSD and the variance for first, second and third order can be generalized as
   $S_{E_kE_k}\left(e^{j\Omega}\right) = S_{EE}\left(e^{j\Omega}\right)\left|1-e^{-j\Omega}\right|^{2k}$ and $\sigma_{E_k}^2 = N_B^2 = \frac{Q^2}{12}\frac{\pi^{2k}}{(2k+1)L^{2k+1}}$.
   Since we know that $Q = \frac{2x_{max}}{2^w} \rightarrow x_{max}^2 = \frac{Q^2 2^{2w}}{2^2}$ and assuming $P_F = \sqrt{3}$ we can derive the SNR for every order of the noise shaping topology according to

$$
\begin{aligned}
\text{SNR} &= 10\log_{10}\left(\frac{\sigma_X^2}{\sigma_E^2}\right) = 10\log_{10}\left(\left(\frac{x_{max}}{P_F}\right)^2 \frac{12(2k+1)L^{2k+1}}{Q^2\pi^{2k}}\right) \\
&= 10\log_{10}\left(\frac{Q^2 2^{2w}}{2^2 3}\frac{12(2k+1)L^{2k+1}}{Q^2\pi^{2k}}\right) = 10\log_{10}\left(2^{2w}\frac{(2k+1)L^{2k+1}}{\pi^{2k}}\right) \\
&= 6.02w + 10\log_{10}\left(\frac{2k+1}{\pi^{2k}}\right) + (2k+1)10\log_{10}(L) \qquad \text{see Eq. (4.25)}
\end{aligned}
$$

3. How are the power spectral density (PSD) and the variance change related to the order of the DSC?

   Solution:

   - In the following we will show how the formula for PSD and variance have been calculated and then derive the general formula for the PSD and the variance based on the order of the noise shaper. The first-order DSC has the following PSD given by

$$
\begin{aligned}
S_{E_1E_1}\left(e^{j\Omega}\right) &= S_{EE}\left(e^{j\Omega}\right)\left|1-e^{-j\Omega}\right|^2 = S_{EE}\left(e^{j\Omega}\right)\left|e^{-j\frac{\Omega}{2}}e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}e^{-j\frac{\Omega}{2}}\right|^2 \\
&= S_{EE}\left(e^{j\Omega}\right)\left|e^{-j\frac{\Omega}{2}}\right|^2\left|e^{j\frac{\Omega}{2}} - e^{-j\frac{\Omega}{2}}\right|^2 = S_{EE}\left(e^{j\Omega}\right)\left|e^{-j\frac{\Omega}{2}}\right|^2\left|2j\sin\left(\frac{\Omega}{2}\right)\right|^2 \\
&= S_{EE}\left(e^{j\Omega}\right)4\sin^2\left(\frac{\Omega}{2}\right).
\end{aligned}
$$

   The general formula for the PSD can be written as

$$
S_{E_kE_k}\left(e^{j\Omega}\right) = S_{EE}\left(e^{j\Omega}\right)\left|1-e^{-j\Omega}\right|^{2k}.
$$

With $\Omega = \frac{2\pi f}{Lf_S}$ and $S_{EE}(f) = \frac{Q^2}{12Lf_S}$ we can give the noise power in audio band (the variance $\sigma_E^2$) as

$$N_B^2 = \sigma_E^2 = S_{EE}(f)2\int_0^{f_B} 4\sin^2\left(\frac{\pi f}{Lf_S}\right) df \qquad \text{see Eq. (4.10).}$$

Using $\int \sin^2(af)\,df = \frac{f}{2} - \frac{1}{4a}\sin(2af)$ and $a = \frac{\pi}{Lf_S}$ we can write

$$N_B^2 = \frac{8Q^2}{12Lf_S}\left[\frac{f}{2} - \frac{1}{4a}\sin(2af)\right]_0^{f_B}.$$

With $\sin(f) = f - \frac{f^3}{3!}$ we get

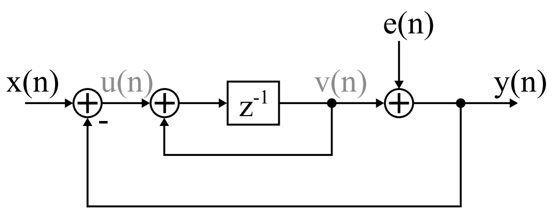
$$\begin{aligned}
N_B^2 &= \frac{8Q^2}{12Lf_S}\left[\frac{f}{2} - \frac{1}{4a}\left(2af - \frac{2^3a^3f^3}{6}\right)\right]_0^{f_B} = \frac{8Q^2}{12Lf_S}\left[\frac{f}{2} - \frac{f}{2} + \frac{a^2f^3}{3}\right]_0^{f_B} \\
&= \frac{8Q^2}{12Lf_S}\left[\frac{a^2f^3}{3}\right]_0^{f_B} \\
&= \frac{Q^2}{12Lf_S}\frac{a^2 2^3 f_B^3}{3} = \frac{Q^2}{12}\frac{\pi^2 2^3 f_B^3}{3\,(L^3 f_S^3)} = \frac{Q^2}{12}\frac{\pi^2(2f_B)^3}{3\,(L^3 f_S^3)} = \frac{Q^2}{12}\frac{\pi^2(f_S)^3}{3\,(L^3 f_S^3)} \\
&= \frac{Q^2}{12}\frac{\pi^2}{3L^3}.
\end{aligned}$$

We can then generalize this formula for all orders $k$ as

$$N_B^2 = \frac{Q^2}{12}\frac{\pi^{2k}}{(2k+1)\,L^{2k+1}}.$$

4. How is noise shaping achieved in an oversampled delta-sigma DA converter?

   Solution:



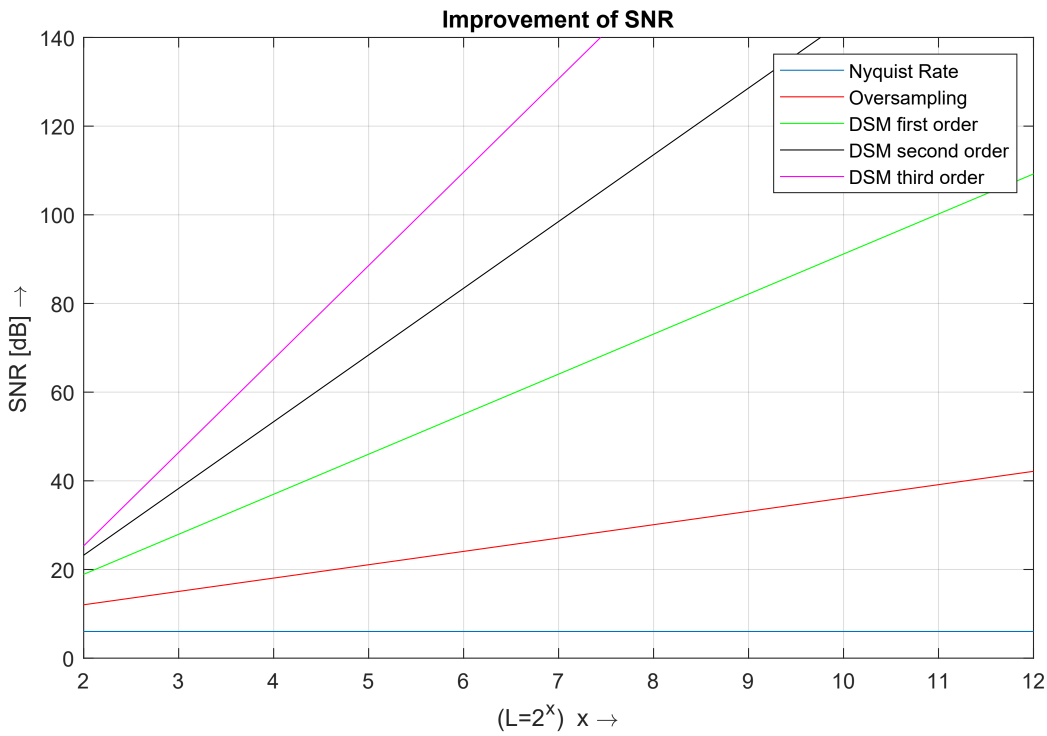

$$\begin{aligned}
y(n) &= v(n) + e(n) \qquad \rightarrow v(n) = y(n) - e(n) \\
u(n) &= x(n) - y(n) \\
v(n) &= u(n-1) + v(n-1) = x(n-1) - y(n-1) + v(n-1)
\end{aligned}$$

$$\begin{aligned}
y(n) &= x(n-1) - y(n-1) + v(n-1) + e(n) \\
&= x(n-1) - y(n-1) + y(n-1) - e(n-1) + e(n) \\
&= x(n-1) + e(n) - e(n-1) \\
Y(z) &= X(z)\cdot z^{-1} + E(z)\cdot\left(1 - z^{-1}\right)
\end{aligned}$$

5. Show the noise-shaping effect (with Matlab plots) of a delta-sigma modulator and how the improvement of the SNR for pure oversampling and delta-sigma modulation is achieved.

   Solution:

   - see Matlab script 'dasp_ex4_2_5'




6. Using the previous Matlab plots, specify which order and oversampling factor $L$ will be needed for a 1-bit delta-sigma converter for SNR = 100 dB.

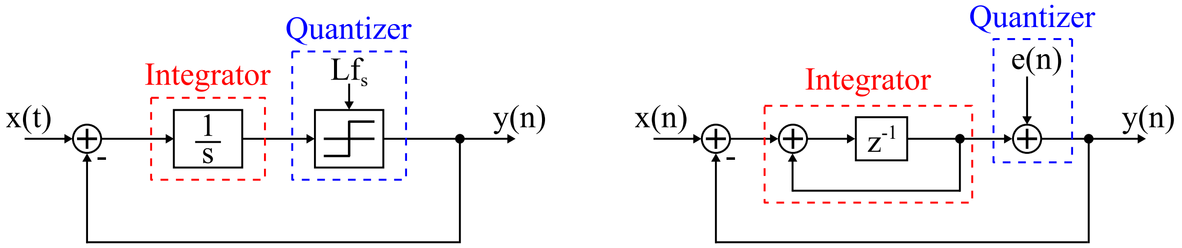
   Solution:

   - By looking on the plot of the previous question. One can se that for a minimum of 100dB SNR one needs:
     - DSM first order: $x = 11 \Rightarrow L = 1048$
     - DSM second order: $x = 8 \Rightarrow L = 256$
     - DSM third order: $x = 6 \Rightarrow L = 64$

7. What is the difference between the delta-sigma modulator in the delta-sigma AD converter and the delta-sigma DA converter?

   Solution:

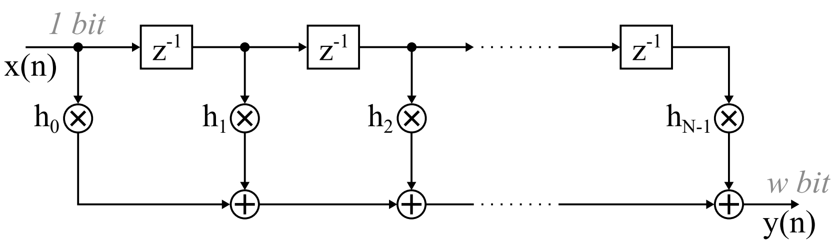
   - They are equal, except for the implementation domain (analog or digital)




8. How do we achieve a $w$-bit signal representation at Nyquist sampling frequency from an oversampled 1-bit signal?

   Solution:

   - Using a digital FIR lowpass filter with coefficients $h_0$... $h_{N-1}$ which have a $w$-bit word length
   - Summation of the filter coefficients $h_0$... $h_{N-1}$ weighted by 1 or 0 of the input sequence leads to a $w$-bit output signal




9. Why do we need to oversample a $w$-bit signal for a delta-sigma DA converter?

   Solution:

   - The oversampling up to a sampling rate $Lf_S$ allows the use of a noise shaper at the oversampling rate.
   - The 1-bit output of the quantizer needs only a following analog lowpass filter to suppress the shaped quantization noise and the image spectra.