DASP3rd Chapter 8 - Exercises

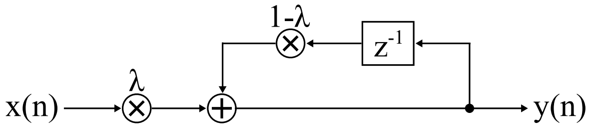
1 Lowpass Filtering for Envelope Detection

Generally, envelope computation is performed by lowpass filtering the input signal's absolute value or its square.

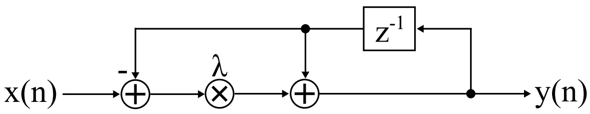
1. Sketch the block diagram of a recursive first-order lowpass $H(z) = \frac{\lambda}{1-(1-\lambda)z^{-1}}$.

   Solution:

   - Of course, there are various possibilities. Most straight-forward is probably the transposed direct-form II

     where it is obvious that the current input sample is weighted with λ and the most recent output sample with $1 - \lambda$.
   - Alternatively, a realization is possible which uses only one multiplier:

     (Note: In the book, z_∞ is used instead of λ, where $\lambda = 1 - z_\infty$.)

2. Sketch its step response. What characteristic measure of the envelope detector can be derived from the step response and how?

   Solution:

   To find the step response, we start with the difference equation $y(n) = \lambda x(n) + (1 - \lambda)y(n-1)$. The step function is defined as:

   $$\epsilon(n) = \begin{cases} 1 & \text{for } n \geq 0 \\ 0 & \text{else} \end{cases}$$

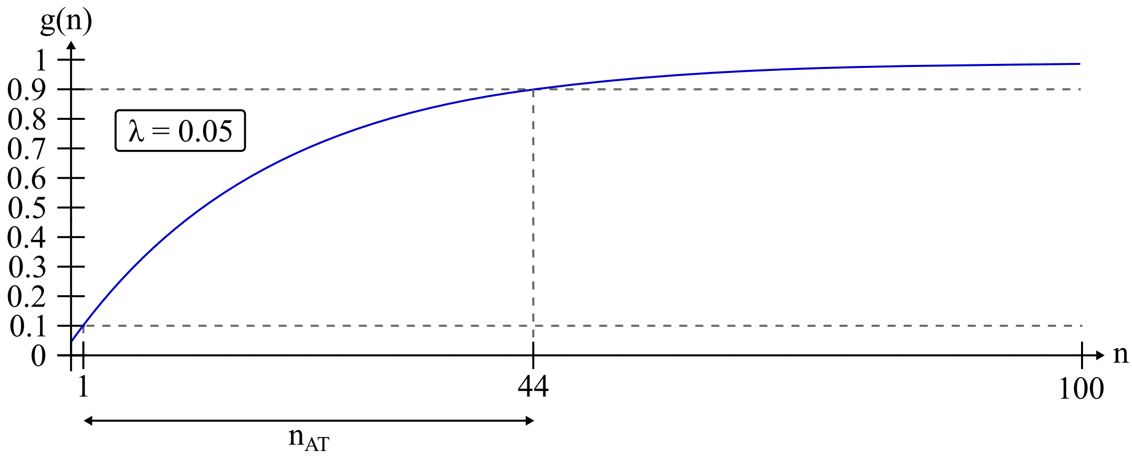
   Now we can evaluate the difference equation with the step function as input for $n = 0, 1, 2$

   $$\begin{aligned} g(0) &= \lambda + (1-\lambda) \cdot 0 \\ g(1) &= \lambda + (1-\lambda)\lambda \\ g(2) &= \lambda + (1-\lambda)\left[\lambda + (1-\lambda)\lambda\right] = \lambda + (1-\lambda)\lambda + (1-\lambda)(1-\lambda)\lambda \\ &\vdots \end{aligned}$$

   Or in a more general way: $g(n) = \lambda \cdot \sum\limits_{k=0}^{n} (1-\lambda)^k$

The geometric series is defined as: $\sum_{k=0}^{n} z^k = \frac{1-z^{n+1}}{1-z}$ with $z = 1 - \lambda$

$$\Rightarrow g(n) = \lambda \sum_{k=0}^{n} (1-\lambda)^k = \lambda \cdot \frac{1-(1-\lambda)^{n+1}}{1-(1-\lambda)} = 1 - (1-\lambda)^{n+1}$$



The step-response is of interest because it represents the response of the envelope detector to a sudden rise in amplitude of the input signal. A typical measure is the time it takes the step-response to rise from 0.1 to 0.9. Ignoring for the moment that n can only take integer values, the attack time in samples is $n_{AT} = n_{0.9} - n_{0.1}$, with $g(n_{0.1}) = 0.1$ and $g(n_{0.9}) = 0.9$. Solving $[1-(1-\lambda)^{n_{0.1}} = 0.1]$ and $[1-(1-\lambda)^{n_{0.9}} = 0.9]$ for $n_{0.1}$ and $n_{0.9}$, we find

$$n_{0.1} = \frac{\ln 0.9}{\ln(1-\lambda)} \quad \text{and} \quad n_{0.9} = \frac{\ln 0.1}{\ln(1-\lambda)},$$

and hence

$$n_{AT} = n_{0.9} - n_{0.1} = \frac{\ln 0.1 - \ln 0.9}{\ln(1-\lambda)} \approx -\frac{2.2}{\ln(1-\lambda)}.$$

To compute λ for a given attack time, we can solve for λ as

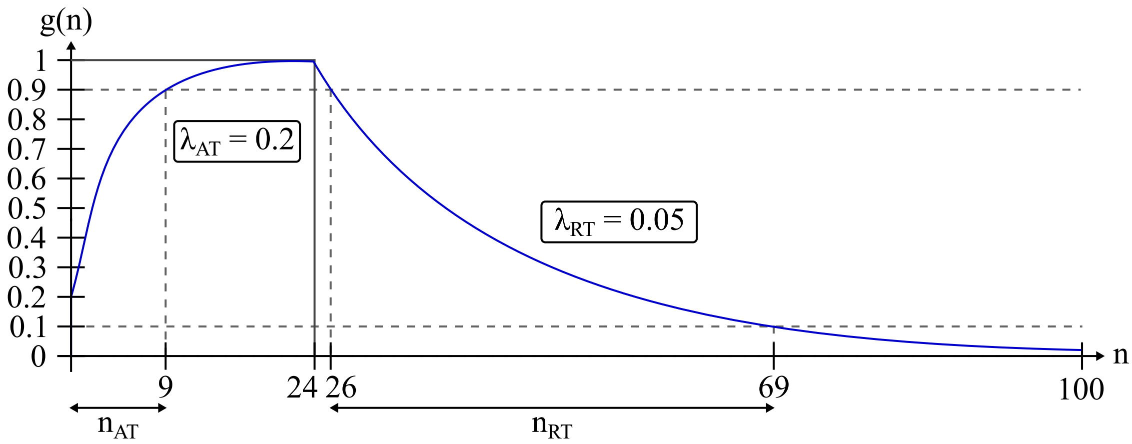
$$\lambda = 1 - e^{\frac{-2.2}{n_{AT}}}$$

and of course we express n_{AT} in terms of an attack-time in seconds T_{AT} as $n_{AT} = f_S \cdot T_{AT}$, where f_S denotes the sampling rate.

3. Typically, the lowpass filter is modified to use a non-constant filter coefficient λ. How does λ depend on the signal? Sketch the response to a rect-signal of the lowpass filter thus modified.

Solution:

Usually, a dynamics processor has to react fast on attacks (rising signal amplitude) to avoid overload, but may be slower for releases (falling amplitude). Therefore, two different coefficients λ_{AT} and λ_{RT} are used, where λ_{AT} is employed when the filter's current input is greater than its most recent output and λ_{RT} otherwise. Typically, $\lambda_{AT} > \lambda_{RT}$, so that $T_{AT} < T_{RT}$.

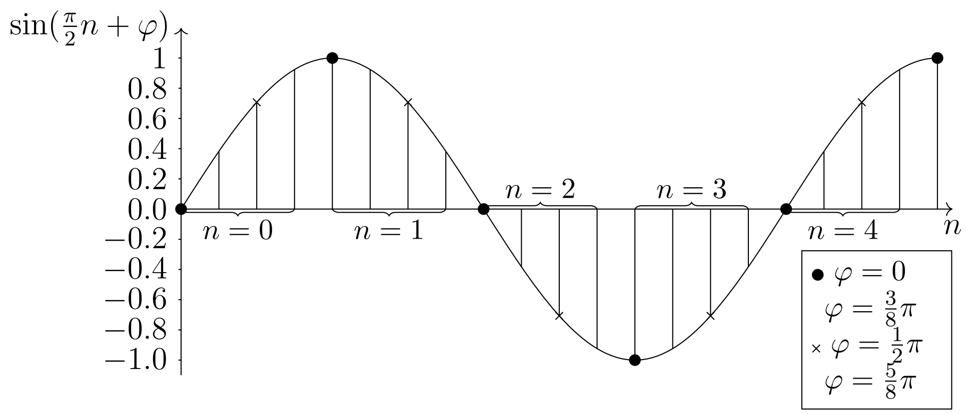


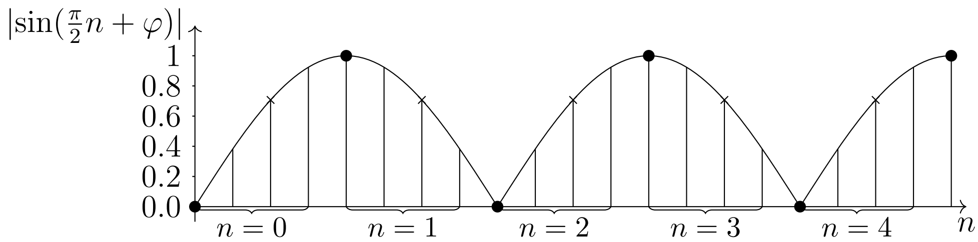
2 Discrete-time Specialties of Envelope Detection

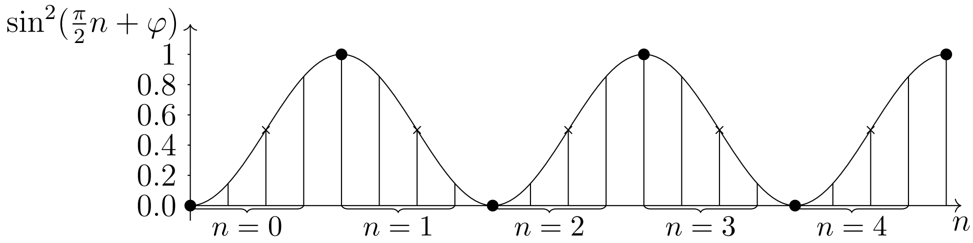
Taking absolute value or squaring are nonlinear operations. Hence care must be taken when using them in discrete-time systems as they introduce harmonics the frequency of which may violate the Nyquist bound. This can lead to unexpected results, as a simple example shall illustrate. Consider the input signal $x(n) = \sin\left(\frac{\pi}{2}n + \varphi\right)$, $\varphi \in [0, 2\pi]$.

1. Sketch $x(n)$, $|x(n)|$, and $x^2(n)$ for different values of φ.

   Solution:







2. Determine the value of the envelope after perfect lowpass filtering, i.e. averaging, $|x(n)|$. Note: As the input signal is periodical, it is sufficient to consider one period, e.g.

$$\bar{x} = \frac{1}{4}\sum_{n=0}^{3}|x(n)|.$$

Solution:

$$\begin{aligned}
\bar{x} &= \tfrac{1}{4}\Bigg(\Big|\underbrace{\sin(\varphi)}_{\geq 0}\Big| + \Big|\underbrace{\sin\left(\tfrac{\pi}{2}+\varphi\right)}_{\geq 0}\Big| + \Big|\underbrace{\sin(\pi+\varphi)}_{\leq 0}\Big| + \Big|\underbrace{\sin\left(\tfrac{3\pi}{2}+\varphi\right)}_{\leq 0}\Big|\Bigg) \\
&= \tfrac{1}{4}\Bigg(\sin(\varphi) + \sin\left(\tfrac{\pi}{2}+\varphi\right) - \underbrace{\sin(\pi+\varphi)}_{=-\sin(\varphi)} - \underbrace{\sin\left(\tfrac{3\pi}{2}+\varphi\right)}_{=-\sin\left(\frac{\pi}{2}+\varphi\right)}\Bigg) \\
&= \tfrac{1}{2}\sin(\varphi) + \tfrac{1}{2}\sin\left(\tfrac{\pi}{2}+\varphi\right) \\
&= \tfrac{1}{2}\sin(\varphi) + \tfrac{1}{2}\Big(\underbrace{\sin\left(\tfrac{\pi}{2}\right)}_{=1}\cos(\varphi) + \underbrace{\cos\left(\tfrac{\pi}{2}\right)}_{=0}\sin(\varphi)\Big) \\
&= \tfrac{1}{2}\left(\sin(\varphi) + \cos(\varphi)\right)
\end{aligned}$$

The minimum $\bar{x}_{min} = \frac{1}{2}$ is reached for $\varphi = 0$ and $\varphi = \frac{\pi}{2}$, the maximum $\bar{x}_{max} = \frac{1}{\sqrt{2}}$ for $\varphi = \frac{\pi}{4}$.

3. Similarly, determine the value of the envelope after averaging $x^2(n)$.

Solution:

$$\begin{aligned}
\bar{x} &= \tfrac{1}{4}\Bigg(\sin^2(\varphi) + \underbrace{\sin^2\left(\tfrac{\pi}{2}+\varphi\right)}_{=\cos^2(\varphi)} + \underbrace{\sin^2(\pi+\varphi)}_{=\sin^2(\varphi)} + \underbrace{\sin^2\left(\tfrac{3\pi}{2}+\varphi\right)}_{=\cos^2(\varphi)}\Bigg) \\
&= \tfrac{1}{2}\left(\sin^2(\varphi) + \cos^2(\varphi)\right) \\
&= \tfrac{1}{2}
\end{aligned}$$

Of course, this is the average power, so to get the root-mean-square, we have to take the square root yielding $\frac{1}{\sqrt{2}}$ as the envelope, independent of the phase φ.

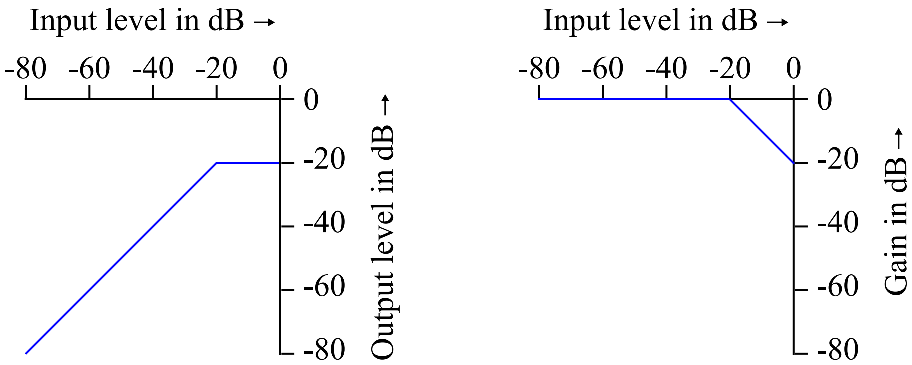
Generally, having an envelope that depends on the phase of the signal is problematic. Therefore, in digital dynamics processors, it is recommended to use root-mean-square envelope detection.

3 Dynamic Range Processors

Sketch the characteristic curves mapping input level to output level and input level to gain for, and describe briefly the application of:

1. limiter;

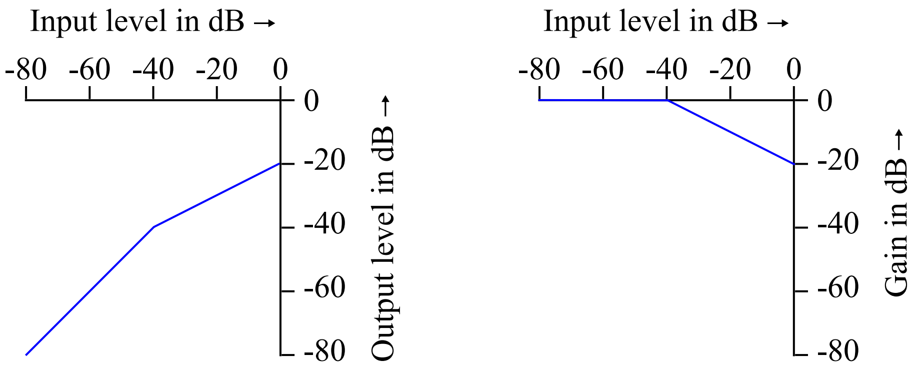
   Solution:



   - Limits the maximum amplitude to avoid overload of successive processing units

2. compressor;

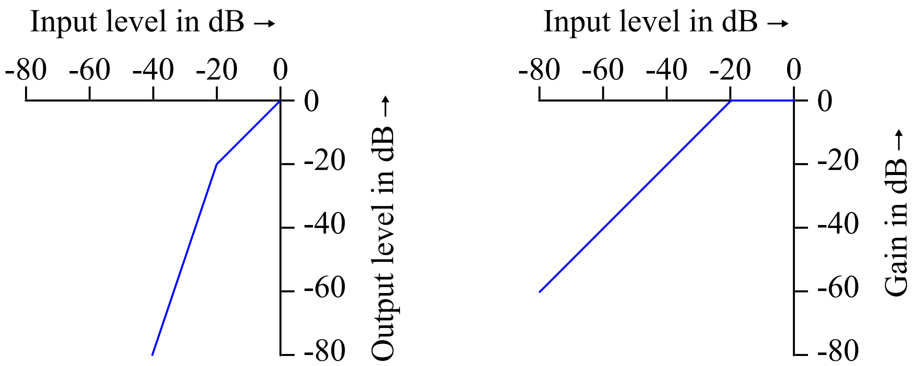
   Solution:



   - Reduces the dynamic range of the signal
   - Can be combined with a constant gain to make a signal louder by amplifying quiet parts

3. expander; and

   Solution:

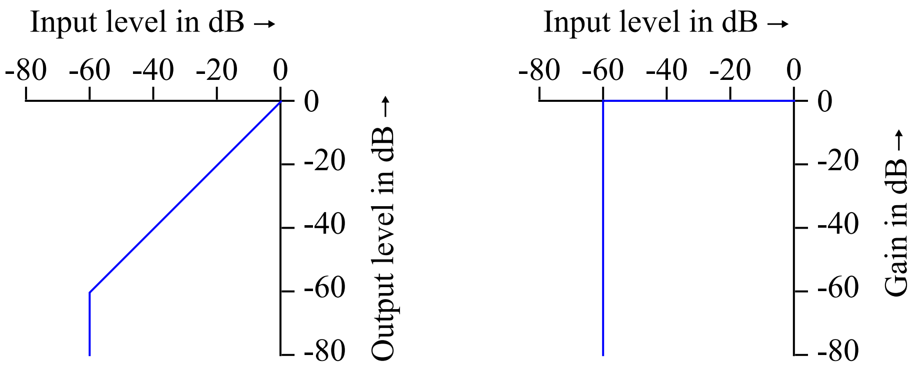


   - Increases the dynamic range of the signal

- Can be used to suppress noise by attenuating quiet parts which mostly consist of noise
- Used in conjunction with a compressor for recording and transmission systems: Compress the signal before recording/transmission to better utilize available dynamic range of the medium, expand the signal upon playback/after transmission to restore original amplitude.

4. noise gate.

   Solution:



   - Suppress noise by muting quiet parts which mostly consist of noise.