DASP3rd Chapter 10 - Exercises

1 Fundamentals

1. What can you say about the output of a memoryless nonlinear system excited with a periodic input?

   Solution:

   The output is also periodic with the same period as the input. However, the nonlinearity will in general introduce additional harmonics, i.e. components at multiples of the input frequency.

2. Let $f(x) = |x|$ be a static nonlinear mapping excited with $x(t) = \cos(\omega_0 t)$. Compute the Fourier coefficients of the resulting output $y(t) = |\cos(\omega_0 t)|$. What can you say about the THD?

   Solution:

   Owing to the periodicity with period $T_0 = \frac{2\pi}{\omega_0}$ and the even symmetry, we can write $y(t)$ as a Fourier series

   $$y(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(\omega_0 kt) \tag{1}$$

   with

   $$a_k = \frac{2}{T_0} \int_{-\frac{T_0}{2}}^{\frac{T_0}{2}} y(t) \cos(\omega_0 kt) dt. \tag{2}$$

   But note that closer inspection reveals that the output actually has period $T_0' = \frac{1}{2} T_0 = \frac{\pi}{\omega_0}$. Therefore its Fourier series can be written as

   $$y(t) = \frac{a_0'}{2} + \sum_{k=1}^{\infty} a_k' \cos(\omega_0' kt) \tag{3}$$

   where $\omega_0' = 2\omega_0$ and by comparison we find

   $$a_k = \begin{cases} a_{k/2}' & \text{for } k \text{ even} \\ 0 & \text{for } k \text{ odd.} \end{cases} \tag{4}$$

   As in particular $a_1 = 0$, we immediately find that for the total harmonic distortion

   $$\mathrm{THD} = \sqrt{\frac{a_2^2 + a_3^2 + \cdots}{a_1^2 + a_2^2 + a_3^2 + \cdots}} \tag{5}$$

   the numerator and denominator become equal and thus $\mathrm{THD} = 1 = 0\,\mathrm{dB}$. For completeness, the non-zero Fourier coefficients are determined as

$$a_{2k} = a'_k = \frac{2}{T'_0}\int_{-\frac{T'_0}{2}}^{\frac{T'_0}{2}} y(t)\cos(\omega'_0 kt)dt = \frac{2\omega_0}{\pi}\int_{-\frac{\pi}{2\omega_0}}^{\frac{\pi}{2\omega_0}} |\cos(\omega_0 t)|\cos(2\omega_0 kt)dt \tag{6}$$

$$= \frac{2\omega_0}{\pi}\int_{-\frac{\pi}{2\omega_0}}^{\frac{\pi}{2\omega_0}} \cos(\omega_0 t)\cos(2\omega_0 kt)dt \tag{7}$$

$$= \frac{\omega_0}{\pi}\int_{-\frac{\pi}{2\omega_0}}^{\frac{\pi}{2\omega_0}} \cos(\omega_0(1-2k)t) + \cos(\omega_0(1+2k)t)dt \tag{8}$$

$$= \frac{1}{\pi}\left[\frac{1}{(1-2k)}\sin(\omega_0(1-2k)t) + \frac{1}{(1+2k)}\sin(\omega_0(1+2k)t)\right]_{-\frac{\pi}{2\omega_0}}^{\frac{\pi}{2\omega_0}} \tag{9}$$

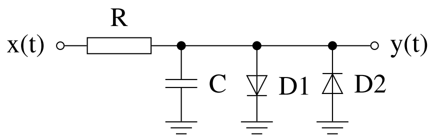
$$= \frac{2}{\pi}\left(\frac{1}{(1-2k)}\sin((1-2k)\frac{\pi}{2}) + \frac{1}{(1+2k)}\sin((1+2k)\frac{\pi}{2})\right) \tag{10}$$

$$= \frac{2}{\pi}\left(\frac{1}{(1-2k)}\cos(k\pi) + \frac{1}{(1+2k)}\cos(k\pi)\right) \tag{11}$$

$$= \frac{4}{\pi(1-(2k)^2)}\cos(k\pi) = \frac{4}{\pi(1-(2k)^2)}(-1)^k. \tag{12}$$

2 Overdrive, Distortion, Clipping

1. Derive the nonlinear differential equation for the first order diode clipper assuming identical diodes. Hint: $i_d = I_s\left(e^{\frac{v_d}{\eta v_t}} - 1\right)$, $\sinh(x) = \frac{e^x - e^{-x}}{2}$



Solution:

$$y(t) = x(t) - R(i_c + i_{d1} - i_{d2}) \tag{13}$$

$$y(t) = x(t) - RC\dot{y}(t) - RI_s\left(e^{\frac{y(t)}{\eta v_t}} - 1 - e^{-\frac{y(t)}{\eta v_t}} + 1\right) \tag{14}$$

$$y(t) = x(t) - RC\dot{y}(t) - RI_s\left(2\sinh\left(\frac{y(t)}{\eta v_t}\right)\right) \tag{15}$$

$$\dot{y}(t) = \frac{1}{RC}\left[x(t) - y(t) - 2RI_s\sinh\left(\frac{y(t)}{\eta v_t}\right)\right] \tag{16}$$

2. What is the difference between soft and hard clipping nonlinearities? How can they be used to create a distortion or overdrive effect?

Solution:

The difference lays in the transition from the linear to the saturation region in the corresponding characteristic curve. A soft clipping nonlinearity has a smooth transition from linear to saturation in contrast to a hard clipping nonlinearity, which transitions abruptly into the saturation region. A hard clipping nonlinearity will produce an higher amount of harmonics in the output signal. A distortion, clipping or overdrive effect can be achieved by combining this nonlinear mapping function with a linear filter responsible for shaping the tone of the effect.

3 Nonlinear Filters

1. How can we extend a linear filter design to give it a more natural sound?

   Solution:

   Nonlinear mapping functions can be inserted into a linear filter resulting into a filter with additional harmonic frequency content.

2. What is the difference to linear filters regarding the stability?

   Solution:

   The pole locations for nonlinear filters are dependent on the state of the filter. Therefore the easiest way to achieve stability is to assure that all pole magnitudes are strictly below 1 for each point in time.

4 Aliasing and its Mitigation

1. Assume a nonlinear system introducing harmonics rolling off with frequency by approximately $1/f$. When operated at a sampling rate of 44.1 kHz, the aliasing distortion is deemed too high. By what factor, approximately, is the aliasing distortion present below 22.05 kHz reduced when doubling the sampling rate?

   Solution:

   At the original smapling rate, the aliasing distortion is dominated by the frequency components just above 22.05 kHz prior to alising. When doubling the sampling frequency, the lowest frequency that gets aliased to 22.05 kHz (or below) is $2 \cdot 44.1\,\mathrm{kHz} - 22.05\,\mathrm{kHz} = 66.15\,\mathrm{kHz} = 3 \cdot 22.05\,\mathrm{kHz}$. Due to the $1/f$ roll-off, the aliasing distortion is therefore reduced by approximately a factor of three.

2. Apply antiderivative antialiasing to the memoryless systems described by the mapping functions

$$f_1(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1/3 \\ \frac{3-(2-3x)^2}{3} & \text{for } 1/3 \leq x \leq 2/3 \\ 1 & \text{for } 2/3 \leq x \end{cases} \tag{17}$$

and

$$f_2(x) = \begin{cases} 2x & \text{for } 0 \leq x \leq 1/2 \\ 1 & \text{for } 1/2 \leq x. \end{cases} \tag{18}$$

Solution:

We first need to determine the antiderivative of the mapping functions. By picking the integration constant such that $F(0) = 0$, we find

$$F_1(x) = \begin{cases} x^2 & \text{for } 0 \leq x \leq 1/3 \\ \frac{1}{27} - \frac{1}{3}x + 2x^2 - x^3 & \text{for } 1/3 \leq x \leq 2/3 \\ x - \frac{7}{27} & \text{for } 2/3 \leq x \end{cases} \tag{19}$$

and

$$F_2(x) = \begin{cases} x^2 & \text{for } 0 \leq x \leq 1/2 \\ x - \frac{1}{4} & \text{for } 1/2 \leq x. \end{cases} \tag{20}$$

Assuming a symmetric setting where $f(-x) = -f(x)$, we further have $F(-x) = F(x)$. The antialiased systems are then given by

$$y(n) = \begin{cases} \frac{1}{2}\left(f(x(n)) + f(x(n-1))\right) & \text{for } x(n) \approx x(n-1) \\ \frac{F(x(n)) - F(x(n-1))}{x(n) - x(n-1)} & \text{otherwise} \end{cases} \tag{21}$$

using either f_1 and F_1 or f_2 and F_2, respectively.

5 Virtual Analog Modeling

1. Give an overview of the three main modeling approaches.

   Solution:

   - Blackbox models: The model construction relies solely on input and output data.
   - Graybox models: Input and output data is used as well as some additional information such as general structure or small signal behaviour.
   - Whitebox models: In this approach the complete information of the modeled circuit is available.

2. How is a circuit element modeled in the Wave domain? Give the connection between wave and Kirchhoff domain.

   Solution:

   A circuit element in the wave domain is described by its incident and reflected wave as well as its port resistance.

   - Incident wave: $a_0 = v_0 + R_0 i_0$
   - Reflected wave: $b_0 = v_0 - R_0 i_0$
   - port voltage: $v_0 = \frac{1}{2}a_0 + \frac{1}{2}b_0$
   - port current: $i_0 = \frac{1}{2R_0}a_0 - \frac{1}{2R_0}b_0$